

Casimir Energy of a Semi-Circular Infinite Cylinder

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Abstract

The Casimir energy of a semi-circular cylindrical shell is calculated by making use of the zeta function technique. This shell is obtained by crossing an infinite circular cylindrical shell by a plane passing through the symmetry axes of the cylinder and by considering only a half of this configuration. All the surfaces, including the cutting plane, are assumed to be perfectly conducting. The zeta functions for scalar massless fields obeying the Dirichlet and Neumann boundary conditions on the semi-circular cylinder are constructed exactly. The sum of these zeta functions gives the zeta function for electromagnetic field in question. The relevant plane problem is considered also. In all the cases the final expressions for the corresponding Casimir energies contain the pole contributions which are the consequence of the edges or corners in the boundaries. This implies that further renormalization is needed in order for the finite physical values for vacuum energy to be obtained for given boundary conditions.

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I. INTRODUCTION

When calculating the ground state energy of a quantum field (the Casimir energy) the main problem is to single out a finite part of the vacuum energy which is initially divergent. Usually for this purpose a subtraction procedure is used with preliminary regularization of the divergent expressions (for example, by introducing ultraviolet cutoff). However in quantum field theory treated with allowance for nontrivial boundary conditions or in the space-time with curvature, a complete renormalization procedure is not formulated explicitly. Therefore, for any specific problem the subtraction procedure should be invented anew. As a result, one succeeds in calculating the Casimir energy only in the problems with known spectra or at least implicitly known spectra. Practically it implies the boundary conditions of high symmetry^{1,2} (parallel plates, sphere, cylinder).

In studies of the Casimir energy it is widely used the zeta function technique^{3,4} which is also referred to as the zeta regularization or zeta renormalization. In fact, the use of the zeta functions, as well as other regularizations, gives only regularized quantities for ground state energy, for effective potential and so on. The necessity to renormalize the expressions obtained in this way certainly remains. However, in some problems the zeta technique gives at once a finite result. Usually the latter is considered to be a renormalized physical answer though generally it is not the case.⁵

When using the zeta regularization in one or another problem, it is desirable to know beforehand whether the finite result can be obtained in this way. In order to answer this question the general analysis of the divergences in the problem at hand should be accomplished. This can be done by calculating the heat kernel coefficients⁶ depending on the geometry of the manifold under consideration. For a large class of situations these coefficients have been obtained.⁷ However there is a number of problems (for example, boundaries with edges or corners) for which no general results regarding the heat trace are known.

In this situation it is undoubtedly worth carrying out, in the framework of the zeta function technique, the calculations of the Casimir energy for new configurations, both the cases being interesting with finite result and with pole contributions left in the final expression for the vacuum energy.

In the present paper we address the calculation of the Casimir energy for boundaries with edges, more precisely, the vacuum energy of electromagnetic field will be calculated for a semi-circular cylindrical shell by making use of the relevant zeta functions. This shell is obtained by crossing an infinite circular cylindrical shell by a plane passing through the symmetry axes of the cylinder. All the surfaces, including the infinite cutting plane, are assumed to be perfectly conducting. Obviously it is sufficient to consider only a half of this configuration (left or right) which we shall refer to as a semi-circular cylindrical shell or, for sake of shortening, as a semi-circular cylinder. The internal boundary value problem for this configuration is nothing else as a semi-cylindrical waveguide. In the theory of waveguides⁸ it is well known that a semi-circular waveguide has the same eigenfrequencies as the cylindrical one but without degeneracy (without doubling) and safe for one frequency series (see below). Notwithstanding the very close spectra, the zeta function technique does not give a finite result for a semi-circular cylinder unlike for a circular one. First the Casimir energy of an infinite perfectly conducting cylindrical shell has been calculated in Ref. 9 by introducing ultraviolet cutoff and recently this result was derived by zeta function technique¹⁰ (see also

Refs. 11–13). As far as we know the asymmetric boundaries such as a semi-circular cylinder have not been considered in the Casimir problem.

The paper is organized as follows. In Sec. II the electromagnetic spectra are considered in details for cylindrical and semi-cylindrical shells. The general solution of the Maxwell equations for boundary conditions chosen is expressed in terms of two scalar functions, longitudinal components of the electric and magnetic Hertz vectors. These scalar functions are the eigenfunctions of the two-dimensional transverse Laplace operator and obey the Dirichlet and Neumann boundary conditions on the conducting surfaces. In Sec. III the spectral zeta function is constructed for the Dirichlet boundary value problem. To this end, the technique is used which has been elaborated before for representing the spectral zeta function, with given eigenfrequency equations, in terms of contour integral. When carrying out the analytic continuation of the zeta function into the physical region, the uniform asymptotic expansion for the modified Bessel functions is used. In the same way, in Sec. IV the zeta function is constructed for a scalar field obeying the Neumann boundary conditions given on the surface of a semi-circular cylindrical shell. The Section V is concerned with the complete zeta function for electromagnetic field with boundary conditions on the semi-circular cylinder. Transition to the relevant two-dimensional problem is also considered here. In the Conclusion (Sec. VI) the results obtained are summarized, and the origin of the pole singularities of the zeta functions at hand and their relation to the respective boundary value problem are briefly discussed.

II. EIGENMODES OF ELECTROMAGNETIC FIELD FOR CIRCULAR AND SEMI-CIRCULAR CYLINDERS

The construction of the solutions to the Maxwell equations with boundary conditions given on closed surfaces proves to be nontrivial problem. Mainly it is due to the vector character of the electromagnetic field^{8,14,15}. In the case of cylindrical symmetry the electric \mathbf{E} and magnetic \mathbf{H} fields are expressed in terms of the electric ($\mathbf{\Pi}'$) and magnetic ($\mathbf{\Pi}''$) Hertz vectors having only one non-zero component

$$\mathbf{\Pi}' = \mathbf{e}_z \Phi(r, \varphi) e^{\pm i k'_z z}, \quad (2.1)$$

$$\mathbf{\Pi}'' = \mathbf{e}_z \Psi(r, \varphi) e^{\pm i k''_z z}. \quad (2.2)$$

Here the cylindrical coordinate system r, φ, z is used with z axes directed along the cylinder axes. The common time-dependent factor $e^{i\omega t}$ is dropped. The scalar functions $\Phi(r, \varphi)$ and $\Psi(r, \varphi)$ are the eigenfunctions of the two-dimensional transverse Laplace operator and meet, respectively, the Dirichlet and Neumann conditions on the boundary $\partial\Gamma$

$$(\nabla_{\perp}^2 + \gamma'^2) \Phi(r, \varphi) = 0, \quad \Phi(r, \varphi)|_{\partial\Gamma} = 0, \quad (2.3)$$

$$(\nabla_{\perp}^2 + \gamma''^2) \Psi(r, \varphi) = 0, \quad \left. \frac{\partial \Psi(r, \varphi)}{\partial n} \right|_{\partial\Gamma} = 0, \quad (2.4)$$

where ∇_{\perp}^2 is the transverse part of the Laplace operator

$$\nabla_{\perp}^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \varphi^2} \quad (2.5)$$

and

$$\gamma'^2 = \omega^2 - k_z'^2, \quad \gamma''^2 = \omega^2 - k_z''^2. \quad (2.6)$$

First we consider a cylindrical shell. In this case the functions $\Phi(r, \varphi)$ and $\Psi(r, \varphi)$ should be 2π -periodic in angular variable φ . As a result the Dirichlet boundary value problem (2.3) has the following unnormalized eigenfunctions (E -modes)

$$\Phi_{nm}(r, \varphi) = \frac{\sin}{\cos}(n\varphi) \begin{cases} J_n(\gamma'_{nm}r), & r < a, \\ H_n^{(1)}(\bar{\gamma}'_{nm}r), & r > a, \end{cases} \quad (2.7)$$

where a is the cylinder radius, $J_n(x)$ are the Bessel functions, $H_n^{(1)}(x)$ are the Hankel functions of the first kind, and γ'_{nm} , $\bar{\gamma}'_{nm}$ stand for the roots of the frequency equations

$$\begin{aligned} J_n(\gamma'_{nm}a) &= 0, & H_n^{(1)}(\bar{\gamma}'_{nm}a) &= 0, \\ n &= 0, 1, 2, \dots, & m &= 1, 2, \dots \end{aligned} \quad (2.8)$$

For the Neumann boundary value problem (2.4) we have the H -modes

$$\Psi_{nm}(r, \varphi) = \frac{\sin}{\cos}(n\varphi) \begin{cases} J_n(\gamma''_{nm}r), & r < a, \\ H_n^{(1)}(\bar{\gamma}''_{nm}r), & r > a, \end{cases} \quad (2.9)$$

where γ''_{nm} and $\bar{\gamma}''_{nm}$ are the roots of the equations

$$\begin{aligned} \left. \frac{d}{dr} J_n(\gamma''_{nm}r) \right|_{r=a} &= 0, & \left. \frac{d}{dr} H_n^{(1)}(\bar{\gamma}''_{nm}r) \right|_{r=a} &= 0, \\ n &= 0, 1, 2, \dots, & m &= 1, 2, \dots \end{aligned} \quad (2.10)$$

As usual, it is assumed that for $r > a$ the eigenfunctions should satisfy the radiation condition.

It is important to note that each root

$$\gamma'_{nm}, \quad \bar{\gamma}'_{nm}, \quad \gamma''_{nm}, \quad \bar{\gamma}''_{nm}, \quad n \geq 1, \quad m \geq 1 \quad (2.11)$$

is doubly degenerate since, according to Eqs. (2.7), (2.9), there are two eigenfunctions which are proportional to either $\sin(n\varphi)$ or $\cos(n\varphi)$. The frequencies with $n = 0$

$$\gamma'_{0m}, \quad \bar{\gamma}'_{0m}, \quad \gamma''_{0m}, \quad \bar{\gamma}''_{0m}, \quad m = 1, 2, \dots \quad (2.12)$$

are independent on φ , and the degeneracy disappears.

For given Hertz vectors $\mathbf{\Pi}'$ and $\mathbf{\Pi}''$ the electric and magnetic fields are constructed by the formulas

$$\begin{aligned} \mathbf{E} &= \nabla \times \nabla \times \mathbf{\Pi}', & \mathbf{H} &= -i\omega \nabla \times \mathbf{\Pi}' & (E\text{-modes}), \\ \mathbf{E} &= i\omega \nabla \times \mathbf{\Pi}'', & \mathbf{H} &= \nabla \times \nabla \times \mathbf{\Pi}'' & (H\text{-modes}). \end{aligned} \quad (2.13)$$

It has been proved¹⁶ that the superposition of these modes gives the general solution to the Maxwell equations in the problem under consideration. An essential merit of using the Hertz polarization vectors is that in this approach the necessity to satisfy the gauge conditions does not arise.

Let us consider a waveguide which is obtained by cutting the infinite cylindrical shell by a plane passing through the symmetry axes of the cylinder (see Fig. 1). All the surfaces are assumed to be perfectly conducting. In this case the boundary value problems (2.3) and (2.4) for the Hertz electric ($\mathbf{\Pi}'$) and magnetic ($\mathbf{\Pi}''$) vectors have the following eigenfunctions

$$\Phi_{nm}(r, \varphi) = \sin(n\varphi) \begin{cases} J_n(\gamma'_{nm} r), & r < a, \\ H_n^{(1)}(\bar{\gamma}'_{nm} r), & r > a, \end{cases} \quad (2.14)$$

$$n = 1, 2, \dots, \quad m = 1, 2, \dots$$

and

$$\Psi_{nm}(r, \varphi) = \cos(n\varphi) \begin{cases} J_n(\gamma''_{nm} r), & r < a, \\ H_n^{(1)}(\bar{\gamma}''_{nm} r), & r > a, \end{cases} \quad (2.15)$$

$$n = 0, 1, 2, \dots, \quad m = 1, 2, \dots$$

The frequencies γ'_{nm} , $\bar{\gamma}'_{nm}$, γ''_{nm} , and $\bar{\gamma}''_{nm}$ are determined by the same equations (2.8) and (2.10). However the new spectral problem has two essential distinctions: i) the frequencies (2.11) are now nondegenerate, and ii) two series of eigenfrequencies

$$\gamma'_{0m}, \quad \bar{\gamma}'_{0m}, \quad m = 1, 2, \dots \quad (2.16)$$

are absent. At first sight one could expect that such a change of the spectrum cannot influence drastically on the ultraviolet behaviour of the relevant spectral density. However, as it will be shown below, the zeta function for a semi-circular cylinder, unlike for a circular one, does not provide a finite answer for the Casimir energy in the problem in question.

In view of all above-mentioned the zeta function for electromagnetic field obeying the boundary conditions on the surface of the semi-circular cylinder is the sum of two zeta functions for scalar massless fields satisfying the Dirichlet and Neumann conditions on the lateral of this cylinder.

III. ZETA FUNCTION FOR DIRICHLET BOUNDARY VALUE PROBLEM

First we consider the Dirichlet boundary conditions. We shall proceed from the following representation for the zeta function in terms of a contour integral for given frequency equations (2.8) with $n = 1, 2, \dots$

$$\zeta_{\text{cyl}}^{\text{D}}(s) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sum_{n=1}^{\infty} \oint_C d\gamma (\gamma^2 + k_z^2)^{-s/2} \frac{d}{d\gamma} \ln \frac{J_n(\gamma a) H_n^{(1)}(\gamma a)}{J_n(\infty) H_n^{(1)}(\infty)}. \quad (3.1)$$

The contour C consists of the imaginary axis $(-i\infty, i\infty)$ and a semi-circle of an infinite radius in the right half-plane of a complex variable γ . The details of obtaining this integral representation can be found in Refs. 17,10,11,18. Contribution into Eq. (3.1) of integration

along a semi-circle of infinite radius vanishes. Therefore upon integration over k_z this formula acquires the form

$$\zeta^{\text{D}}(s) = C(s) \sum_{n=1}^{\infty} \int_0^{\infty} dy y^{1-s} \frac{d}{dy} \ln [2y I_n(y) K_n(y)] \quad (3.2)$$

with

$$C(s) = \frac{a^{s-1}}{2\sqrt{\pi}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{3-s}{2}\right)}. \quad (3.3)$$

In order to accomplish the analytic continuation of (3.2) into the physical region including the point $s = -1$, we shall use the uniform asymptotic expansion for the modified Bessel functions¹⁹

$$\begin{aligned} \ln [2yn I_n(ny) K_n(ny)] &= \ln(yt) + \frac{t^2}{8n^2}(1 - 6t^2 + 5t^4) \\ &+ \frac{t^4}{64n^4}(13 - 284t^2 + 1062t^4 - 1356t^6 + 565t^8) + O(n^{-6}), \end{aligned} \quad (3.4)$$

where $t = 1/\sqrt{1+y^2}$. Following the usual procedure applied in the analogous calculations,²⁰⁻²³ we add and subtract in the integrand in Eq. (3.2) the first two terms of the asymptotic expansion (3.3). After that we combine all the terms there in the following way

$$\zeta_{\text{cyl}}^{\text{D}}(s) = C(s) [Z_1(s) + Z_2(s) + Z_3(s)], \quad (3.5)$$

$$Z_1(s) = \frac{1}{2} \sum_{n=1}^{\infty} n^{1-s} \int_0^{\infty} dy y^{1-s} \frac{d}{dy} \ln \left(\frac{y^2}{1+y^2} \right), \quad (3.6)$$

$$Z_2(s) = \frac{1}{8} \sum_{n=1}^{\infty} n^{-1-s} \int_0^{\infty} dy y^{1-s} \frac{d}{dy} [t^2(1 - 6t^2 + 5t^4)], \quad (3.7)$$

$$\begin{aligned} Z_3(s) &= \sum_{n=1}^{\infty} n^{1-s} \int_0^{\infty} dy y^{1-s} \frac{d}{dy} \left[\ln(2yn I_n(ny) K_n(ny)) \right. \\ &\quad \left. - \ln \frac{y}{\sqrt{1+y^2}} - \frac{t^2(1 - 6t^2 + 5t^4)}{8n^2} \right]. \end{aligned} \quad (3.8)$$

Analytic continuation of the function $Z_1(s)$ into vicinity of the point $s = -1$ can be accomplished in the same way as it has been done in Ref. 18. Therefore we write here only the final result of this continuation

$$Z_1(s) = \frac{1}{2} \zeta(s-1) \Gamma\left(\frac{3-s}{2}\right) \sum_{m=1}^{\infty} \frac{\Gamma\left(m - \frac{1-s}{2}\right)}{m\Gamma(m)}. \quad (3.9)$$

The integral in Eq. (3.6) converges when $-1 < \text{Re } s < 3$, and the sum over n is finite for $\text{Re } s > 0$. Thus, the regions, where the integral and the sum exist, overlap, and this formula can be used for constructing the analytic continuation needed. For this aim we substitute the sum by the Riemann zeta function

$$\sum_{n=1}^{\infty} n^{-1-s} = \zeta(s+1) \quad (3.10)$$

and define the integral as an analytic function by making use of the formula²⁴

$$\int_0^{\infty} dy y^{1-s} \frac{d}{dy} t^{2(\rho-1)} = (1-\rho) \frac{\Gamma\left(\frac{3-s}{2}\right) \Gamma\left(\rho - \frac{3-s}{2}\right)}{\Gamma(\rho)}, \quad 3 - 2 \operatorname{Re} \rho < \operatorname{Re} s < 3. \quad (3.11)$$

In view of the poles of the gamma functions on the right-hand side of this relation, the integral on the left-hand side of it is well defined, as a function of the complex variable s , only in the region indicated in Eq. (3.11). Doing the analytic continuation of this integral we define it outside this region also by this equation, keeping in mind that the gamma functions involved should be treated as the analytic functions over all the plane of the complex variable s save for the known poles. This gives

$$Z_2(s) = \frac{1}{8} \zeta(s+1) \Gamma\left(\frac{3-s}{2}\right) \Gamma\left(\frac{1+s}{2}\right) \left[-1 + 3(1+s) - \frac{5}{8}(3+s)(1+s)\right]. \quad (3.12)$$

In order to investigate the convergence of the integral entering in Eq. (3.7) it makes sense to substitute in the integrand the logarithmic function by expansion (3.3). After that it is easy to be convinced that the integral under consideration converges when $-3 < \operatorname{Re} s < 3$. The sum over n in this formula is finite for $\operatorname{Re} s > -2$. Hence, the function $Z_3(s)$ is an analytic function without singularities in the domain $-2 < \operatorname{Re} s < 3$. It is quiet enough for our purpose, and the analytic continuation is unnecessary.

Summarizing we conclude that Eqs. (3.3), (3.5), (3.8), (3.9), and (3.12) afford the analytic continuation needed and define the zeta function $\zeta^D(s)$ as an analytic function in the region including the point $s = -1$.

Now we are able to calculate the value of the zeta function $\zeta^D(s)$ at the point $s = -1$. For the coefficient $C(s)$ in Eq. (3.3) we have

$$C(-1) = -\frac{1}{4\pi a^2}. \quad (3.13)$$

From Eq. (3.9) it follows that

$$Z_1(-1) = \frac{1}{2} \lim_{s \rightarrow -1} \zeta(s-1) \left[\Gamma\left(\frac{1+s}{2}\right) + \sum_{m=2}^{\infty} \frac{1}{m(m-1)} \right]. \quad (3.14)$$

With allowance for the relations

$$\Gamma(x) = \frac{1}{x} - \gamma + O(x), \quad \sum_{m=2}^{\infty} \frac{1}{m(m-1)} = 1, \quad \zeta(-2) = 0, \quad (3.15)$$

where γ is the Euler constant, $\gamma = 0.577215\dots$, one derives

$$\begin{aligned} Z_1(-1) &= \lim_{s \rightarrow -1} \frac{1}{2} \left[\zeta(-2) + \zeta'(-2)(s+1) + O((s+1)^2) \right] \left[\frac{2}{s+1} - \gamma + O(s+1) \right] \\ &= \zeta'(-2) = -0.030448. \end{aligned} \quad (3.16)$$

Using the values of the Riemann zeta function and its derivative at the origin

$$\zeta(0) = -\frac{1}{2}, \quad \zeta'(0) = -\frac{1}{2} \ln(2\pi)$$

and taking into account the behaviour of the gamma function near zero (see Eq. (3.15)) we deduce from Eq. (3.12)

$$\begin{aligned} Z_2(-1) &= \frac{1}{8} \lim_{s \rightarrow -1} \left[\zeta(0) + \zeta'(0)(s+1) + O((s+1)^2) \right] \\ &\quad \times \left[\frac{2}{s+1} - \gamma + \mathcal{O}(s+1) \right] \cdot \left[-1 + \frac{7}{4}(s+1) \right] \\ &= -\frac{7}{32} - \frac{\gamma}{16} + \frac{1}{8} \ln(2\pi) + \frac{1}{8} \frac{1}{s+1} \Big|_{s \rightarrow -1}. \end{aligned} \quad (3.17)$$

When calculating $Z_3(-1)$ we shall use Eq. (3.8) for several first values of n , $n \leq n_0$ and for $n > n_0$ we substitute the asymptotic expansion (3.4) into (3.8) with the result

$$\begin{aligned} Z_3^{\text{as}}(s) &= \frac{1}{64} \left(\sum_{n=n_0+1}^{\infty} n^{-3-s} \right) \int_0^{\infty} dy y^{1-s} \frac{d}{dy} \left[t^4 (13 - 284t^2 + 1062t^4 - 1356t^6 + 565t^8) \right] \\ &= \frac{1}{64} \left(\sum_{n=n_0+1}^{\infty} n^{-3-s} \right) \Gamma\left(\frac{3-s}{2}\right) \left[-13 \Gamma\left(\frac{3+s}{2}\right) + 142 \Gamma\left(\frac{5+s}{2}\right) - \frac{532}{3} \Gamma\left(\frac{7+s}{2}\right) \right. \\ &\quad \left. + \frac{113}{2} \Gamma\left(\frac{9+s}{2}\right) - \frac{113}{24} \Gamma\left(\frac{11+s}{2}\right) \right]. \end{aligned} \quad (3.18)$$

The value of n_0 should be chosen so as to provide the accuracy needed. This algorithm with $n_0 = 6$ gives for $Z_3(-1)$

$$Z_3(-1) = 0.022806 \quad (3.19)$$

Summing up Eqs. (3.16), (3.17), and (3.19) we obtain

$$\begin{aligned} \zeta^{\text{D}}(-1) &= -\frac{1}{4\pi a^2} \left(-\frac{7}{32} + 0.022806 - \frac{\gamma}{16} + \frac{1}{8} \ln(2\pi) + \zeta'(-2) + \frac{1}{8} \frac{1}{s+1} \Big|_{s \rightarrow -1} \right) \\ &= \frac{1}{a^2} \left(0.000523 - 0.009947 \frac{1}{s+1} \Big|_{s \rightarrow -1} \right). \end{aligned} \quad (3.20)$$

Thus the zeta function $\zeta^{\text{D}}(s)$ has a pole at the point $s = -1$, therefore it does not give the finite (renormalized) value for the respective Casimir energy

$$E^{\text{D}} = \frac{1}{2} \zeta^{\text{D}}(-1). \quad (3.21)$$

It implies that further renormalization is required.

IV. ZETA FUNCTION FOR NEUMANN BOUNDARY VALUE PROBLEM

When constructing the zeta function for the boundary value problem (2.4) with $\partial\Gamma$ being a semi-circular infinite cylinder, we shall again proceed from the frequency equations (now from Eq. (2.10)). It should be taken into account that all these roots are not degenerate. Therefore we can write analogously to Eq. (3.1)

$$\zeta^N(s) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{dk_z}{2\pi} \sum_{n=0}^{\infty} \oint_C d\gamma (\gamma^2 + k_z^2)^{-s/2} \frac{d}{d\gamma} \ln \frac{J'_n(\gamma a) H_n^{(1)'}(\gamma a)}{J'_n(\infty) H_n^{(1)'}(\infty)}. \quad (4.1)$$

The contour C is the same as in Eq. (3.1) and the prime on the Bessel and Hankel functions denotes differentiation with respect to the entire argument.

The product of the derivatives of the modified Bessel functions $I'_n(z)K'_n(z)$ has the following asymptotics when n is fixed and $|z|$ is large¹⁹

$$I'_n(z)K'_n(z) = -\frac{1}{2z} \left[1 + \frac{4n^2 - 3}{2(2z)^2} + \frac{(4n^2 - 1)(4n^2 - 45)}{8(2z)^4} + O(z^{-6}) \right]. \quad (4.2)$$

Taking this into account in calculation of the denominator in Eq. (4.1), we obtain for $\zeta^N(s)$ upon integration over k_z

$$\zeta^N(s) = C(s) \sum_{n=0}^{\infty} \int_0^{\infty} dy y^{1-s} \frac{d}{dy} \ln [-2y I'_n(y) K'_n(y)] \quad (4.3)$$

with the same function $C(s)$ as in Eq. (3.3).

Further we shall use the uniform asymptotic expansion for the derivatives of the Bessel functions¹⁹

$$\begin{aligned} \ln [-2yn I'_n(ny) K'_n(ny)] &= -\ln(yt) + \frac{t^2}{8n^2}(-3 + 10t^2 - 7t^4) + \frac{t^4}{n^4} \left(-\frac{27}{64} \right. \\ &\quad \left. + \frac{109}{16}t^2 - \frac{733}{32}t^4 + \frac{441}{16}t^6 - \frac{707}{64}t^8 \right) + \mathcal{O}(n^{-6}). \end{aligned} \quad (4.4)$$

In order to render the integral in the term with $n = 0$ in Eq. (4.3) convergent we add and subtract the second term from the asymptotics (4.4). For $n \geq 1$ in Eq. (4.3) we add and subtract in respective integrands the first two terms of the asymptotic expansion (4.4). After that we combine all the terms in the following way

$$\zeta^N(s) = C(s) [V_0(s) + V_1(s) + V_2(s) + V_3(s)], \quad (4.5)$$

$$V_0(s) = \int_0^{\infty} dy y^{1-s} \frac{d}{dy} \left\{ \ln [-2y I'_0(y) K'_0(y)] - \frac{t^2}{8}(-3 + 10t^2 - 7t^4) \right\}, \quad (4.6)$$

$$V_1(s) = -\frac{1}{2} \sum_{n=1}^{\infty} n^{1-s} \int_0^{\infty} dy y^{1-s} \frac{d}{dy} \ln \left(\frac{y^2}{1 + y^2} \right) = -Z_1(s), \quad (4.7)$$

$$V_2(s) = \frac{1}{8} \left(\sum_{n=1}^{\infty} n^{-1-s} + 1 \right) \int_0^{\infty} dy y^{1-s} \frac{d}{dy} [t^2(-3 + 10t^2 - 7t^4)], \quad (4.8)$$

$$\begin{aligned} V_3(s) &= \sum_{n=1}^{\infty} n^{1-s} \int_0^{\infty} dy y^{1-s} \frac{d}{dy} \left\{ \ln [-2yn I'_n(ny) K'_n(ny)] \right. \\ &\quad \left. + \ln(yt) - \frac{t^2}{8n^2}(-3 + 10t^2 - 7t^4) \right\}. \end{aligned} \quad (4.9)$$

Taking into account the behaviour of the product $I'_0(y)K'_0(y)$ at the origin and at infinity

$$\begin{aligned} -2yI'_0(y)K'_0(y) &= y + \frac{1}{8}(-1 + 4y - 4\ln 2 + \ln y)y^3 + O(y^5 \ln y), \\ -2yI'_0(y)K'_0(y) &= 1 - \frac{3}{8y^2} + \frac{45}{128y^4} + O(y^{-6}) \end{aligned} \quad (4.10)$$

it is easy to show that Eq. (4.6) defines $V_0(s)$ as an analytic function in the region $-3 < \text{Re } s < 1$. Under this condition the integration by parts can be done here

$$V_0(s) = -(1-s) \int_0^\infty dy y^{-s} \left\{ \ln[-2yI'_0(y)K'_0(y)] - \frac{t^2}{8}(-3 + 10t^2 - 7t^4) \right\}. \quad (4.11)$$

The function $V_1(s)$ differs only in sign of the function $Z_1(s)$ from the proceeding Section. The integral in Eq. (4.7) is convergent when $-1 < \text{Re } s < 3$. The sum over n in this formula is finite when $\text{Re } s > 0$. Thus the regions, where the integral and the sum exist, overlap and this formula can be used for constructing the analytic continuation needed by making use of the substitutions (3.10) and (3.11). Substituting the sum in Eq. (4.8) by the Riemann zeta function and doing the integration according to Eq. (3.11) one obtains

$$V_2(s) = \frac{1}{8}[\zeta(1+s) + 1]\Gamma\left(\frac{3-s}{2}\right)\Gamma\left(\frac{1+s}{2}\right)\left[3 - 5(1+s) + \frac{7}{8}(1+s)(3+s)\right]. \quad (4.12)$$

The convergence of the integral in Eq. (4.9) can be determined in the same line as it has been done for the function $Z_3(s)$ in the preceding Section. This integral converges when $-3 < \text{Re } s < 3$, and the sum encountered here is finite for $\text{Re } s > -2$. Hence there is no need to do analytic continuation for $V_3(s)$.

Finally the zeta function $\zeta^N(s)$ for the massless scalar field obeying the Neumann boundary conditions on a semi-circular cylinder is determined explicitly by Eqs. (4.5), (4.7), (4.11), and (4.12) in a finite domain of the complex plane s containing the closed interval of the real axis $-1 \leq \text{Re } s \leq 0$.

Now we turn to the calculation of the value of the function $\zeta^N(s)$ at the point $s = -1$. Integration in Eq. (4.11) gives

$$\begin{aligned} V_0(-1) &= -2 \int_0^\infty dy y \left\{ \ln[-2yI'_0(y)K'_0(y)] + \frac{3}{8}t^2 \right\} + \frac{13}{16} \\ &= 2 \cdot 0.475215 + 0.8123 = 1.76393. \end{aligned} \quad (4.13)$$

From Eqs. (4.7) and (3.16) it follows that

$$V_1(-1) = -Z_1(-1) = -\zeta'(-2) = 0.03044. \quad (4.14)$$

Developing the functions $\zeta(1+s)$ and $\Gamma((1+s)/2)$ in Eq. (4.12) near the point $s = -1$ one obtains

$$\begin{aligned} V_2(-1) &= \frac{1}{8} \left[\zeta(0) + \zeta'(0)(s+1) + 1 + O((s+1)^2) \right] \cdot \left[\frac{2}{1+s} - \gamma + O(s+1) \right] \\ &\times \left[3 - \frac{13}{4}(s+1) \right] = -\frac{13}{32} - \frac{3}{16}\gamma + \frac{3}{4}\zeta'(0) + \frac{3}{8} \frac{1}{s+1} \Big|_{s \rightarrow -1}. \end{aligned} \quad (4.15)$$

When calculating $V_3(s)$ for $s = -1$ numerically we cannot use the method applied in the preceding section because it requires now to take into account the next terms in the uniform asymptotic expansion (4.4). Instead of this we calculate numerically the first 30 terms in the sum (4.9) with the result²⁵

$$V_3(-1) = -0.04366. \quad (4.16)$$

Substituting in Eq. (4.9) the logarithm by its uniform asymptotic expansion (4.4) we derive a rough estimation for $V_3(s)$ without numerical integration

$$\begin{aligned} V_3^{\text{as}}(s) &= \zeta(3+s) \int_0^\infty dy y^{1-s} \frac{d}{dy} \left[t^4 \left(-\frac{27}{64} + \frac{109}{16}t^2 - \frac{733}{32}t^4 + \frac{441}{16}t^6 - \frac{707}{64}t^8 \right) \right] \\ &= \zeta(3+s) \Gamma\left(\frac{3-s}{2}\right) \left[\frac{27}{64} \Gamma\left(\frac{3+s}{2}\right) - \frac{109}{32} \Gamma\left(\frac{5+s}{2}\right) + \frac{733}{192} \Gamma\left(\frac{7+s}{2}\right) \right. \\ &\quad \left. - \frac{441}{384} \Gamma\left(\frac{9+s}{2}\right) + \frac{707}{7680} \Gamma\left(\frac{11+s}{2}\right) \right]. \end{aligned} \quad (4.17)$$

For $s = -1$ it gives

$$V_3^{\text{as}}(-1) = -\frac{839}{2^6 \cdot 3 \cdot 5} \zeta(2) = -\frac{839 \pi^2}{960 \cdot 6} = -1.43760, \quad (4.18)$$

that is very far from Eq. (4.16) having only the right sign.

Summing up V_i , $i = 0, 1, 2, 3$ with allowance for Eq. (3.13) we arrive at the final result

$$\begin{aligned} \zeta^{\text{N}}(-1) &= -\frac{1}{4\pi a^2} \left[\frac{13}{32} + 0.95043 - \zeta'(-2) - \frac{3}{16}\gamma \right. \\ &\quad \left. - \frac{3}{8} \ln(2\pi) - 0.04366 + \frac{3}{8} \frac{1}{s+1} \Big|_{s \rightarrow -1} \right] \\ &= \frac{1}{a^2} \left(-0.04345 - 0.0298 \frac{1}{s+1} \Big|_{s \rightarrow -1} \right). \end{aligned} \quad (4.19)$$

Thus both the zeta functions for Dirichlet and Neumann boundary conditions have the pole at the point $s = -1$. Hence an additional renormalization is needed in order for a finite physical value of the relevant Casimir energies to be obtained.

V. VACUUM ENERGY OF ELECTROMAGNETIC FIELD WITH BOUNDARY CONDITIONS ON A SEMI-CIRCULAR CYLINDER

Analysis of the spectral problem for the electromagnetic field with boundary conditions on a semi-circular cylinder (see Sec. II) implies that the zeta function for this field is the sum of two zeta functions calculated in the preceding Sections

$$\zeta^{\text{EM}}(s) = \zeta^{\text{D}}(s) + \zeta^{\text{N}}(s). \quad (5.1)$$

Substitution of Eqs. (3.20) and (4.17) into Eq. (5.1) gives

$$\begin{aligned}\zeta^{\text{EM}}(-1) &= -\frac{1}{4\pi a^2} \left[\frac{1}{4} + 0.95043 - \frac{\gamma}{4} - \frac{1}{4} \ln(2\pi) - 0.04366 + \frac{1}{2} \frac{1}{s+1} \Big|_{s \rightarrow -1} \right] \\ &= \frac{1}{a^2} \left(-0.04401 - 0.03978 \frac{1}{s+1} \Big|_{s \rightarrow -1} \right).\end{aligned}\quad (5.2)$$

In both the zeta functions $\zeta^{\text{D}}(s)$ and $\zeta^{\text{N}}(s)$ the pole terms have the same sign. As a result the pole contribution in the sum (5.1) retains. Thus, the situation here proves to be analogous to that when calculating, in the framework of zeta technique, the vacuum energy for spheres in spaces of even dimensions.^{21–23}

As was noted above, we have derived the exact expressions for the zeta functions in question which determine these functions as analytic functions of the complex variable s in a finite region of the plane s containing the closed interval of the real axis $-1 \leq \text{Re } s \leq 0$. It enables one to construct in a straightforward way the spectral zeta functions for relevant boundary value problem on the plane by making use of the relation¹⁸

$$\zeta_{\text{s-cir}}(s) = 2\sqrt{\pi} \frac{\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta_{\text{s-cyl}}(s), \quad (5.3)$$

where $\zeta_{\text{s-cir}}$ is the Dirichlet or the Neumann zeta function for a semi-circle, and $\zeta_{\text{s-cyl}}$ is the respective zeta function for semi-circular cylinder. We shall use this relation for calculating the values $\zeta_{\text{s-cir}}^{\text{D}}(-1)$ and $\zeta_{\text{s-cir}}^{\text{N}}(-1)$ which determine the vacuum energy of the massless scalar fields defined on the half-plane and obeying, respectively, the Dirichlet or Neumann boundary conditions on a semi-circle (see Fig. 1).

For $\zeta_{\text{s-cir}}^{\text{D}}(-1)$ we get from Eqs. (5.3), (3.4), and (3.2)

$$\zeta_{\text{s-cir}}^{\text{D}}(-1) = -\frac{1}{\pi a} \sum_{i=1}^3 Z_i(0). \quad (5.4)$$

When $s = 0$ integration in Eq. (3.6) can be done explicitly with the result

$$Z_1(0) = -\zeta(-1) \int_0^\infty dy \ln \frac{y}{\sqrt{1+y^2}} = \frac{1}{12} \left(-\frac{\pi}{2} \right) = -\frac{\pi}{24}. \quad (5.5)$$

From Eq. (3.12) it follows that

$$Z_2(0) = \frac{\pi}{128} \left(\frac{1}{s} \Big|_{s \rightarrow 0} + \gamma \right). \quad (5.6)$$

Numerical integration in Eq. (3.8) with $s = 0$ gives

$$Z_3(0) = -0.00304. \quad (5.7)$$

Summing up Eqs. (5.5), (5.6), and (5.7) we arrive at the result

$$\begin{aligned}\zeta_{\text{s-cir}}^{\text{D}}(-1) &= \frac{1}{a} \left(\frac{1}{24} - \frac{\gamma}{128} + 0.00097 - \frac{1}{128} \frac{1}{s} \Big|_{s \rightarrow 0} \right) \\ &= \frac{1}{a} \left(0.038127 - \frac{1}{128} \frac{1}{s} \Big|_{s \rightarrow 0} \right).\end{aligned}\quad (5.8)$$

Following the same way one can write

$$\zeta_{\text{s-cir}}^{\text{N}}(-1) = -\frac{1}{\pi a} \sum_{n=0}^3 V_i(0). \quad (5.9)$$

Using Eq. (4.11) one gets

$$\begin{aligned} V_0(0) &= -\int_0^\infty dy \left\{ \ln[-2yI'_0(y)K'_0(y)] + \frac{3}{8}t^2 \right\} - \frac{\pi}{64} \\ &= 0.475175 - \frac{\pi}{64}. \end{aligned} \quad (5.10)$$

From Eqs. (4.7) and (5.5) it follows that

$$V_1(0) = -Z_1(0) = \frac{\pi}{24}. \quad (5.11)$$

Equation (4.12) gives

$$V_2(0) = \frac{5\pi}{128} \left(1 + \gamma + \frac{1}{s} \Big|_{s \rightarrow 0} \right). \quad (5.12)$$

For $V_3(0)$ numerical integration in Eq. (4.9) with $s = 0$ gives

$$V_3(0) = -0.005659. \quad (5.13)$$

Finally, we have

$$\begin{aligned} \zeta_{\text{s-cir}}^{\text{N}}(-1) &= \frac{1}{a} \left[-0.15132 - \frac{5}{128} \left(\gamma + \frac{5}{3} \right) + 0.00180 - \frac{5}{128} \frac{1}{s} \Big|_{s \rightarrow 0} \right] \\ &= \frac{1}{a} \left(-0.237103 - 0.0124 \frac{1}{s} \Big|_{s \rightarrow 0} \right). \end{aligned} \quad (5.14)$$

Both the functions $\zeta_{\text{s-cir}}^{\text{D}}(s)$ and $\zeta_{\text{s-cir}}^{\text{N}}(s)$ have the pole at the point $s = -1$ with the coefficients of the same (negative) sign. For electromagnetic field defined on a plane the boundary conditions reduce to the Neumann conditions. Hence the relevant zeta function is $\zeta_{\text{s-cir}}^{\text{N}}(s)$.

VI. CONCLUSION

In the paper the spectral zeta functions are constructed for massless scalar fields obeying the Dirichlet and Neumann boundary conditions on a semi-circular infinite cylinder. Proceeding from this, the zeta function for electromagnetic field is also derived for such a configuration. In all three cases, the final expressions for the relevant Casimir energy contains the pole contribution. Hence for obtaining the physical result an additional renormalization is needed.

It is essential that for the zeta functions $\zeta(s)$ the exact formulas are derived which determine these functions in a finite region of the complex variable s but not at the vicinity of one point $s = -1$. This allowed one to get in a straightforward way the zeta functions for

the two dimensional (plane) version of the boundary value problem at hand, i.e. the zeta functions for scalar fields defined on a half-plane and obeying the Dirichlet and Neumann boundary conditions on a semi-circle. In this case the final expression for the vacuum energy contains the pole contributions also.

Notwithstanding the spectrum of a semi-circular cylinder is very close to the spectrum of circular one, the zeta function technique does not give a finite value for vacuum energy in the first case and does for the second configuration. In a recent paper²⁶ the divergences found in our consideration are attributed to the existence of edges or corners in the boundaries under investigation.

Closing, it is worth noting that, as far as we know, such boundary conditions with asymmetric geometry (semi-circular cylinder) has been considered in the Casimir problem for the first time.

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FIGURES

FIG. 1. The cross section of an infinite semi-circular cylindrical shell of radius a . All the surfaces (bold-faced lines) are assumed to be perfectly conducting. At the same time this picture presents the two-dimensional (plane) version of the problem under consideration, i.e., the semi-circular boundaries for massless fields defined on the plane.

